d. Consider the function:

$$f(x) = \frac{\sin(\frac{x}{3})}{2x}.$$

(a) (5 points) Find a Taylor series centered at $a = 0$ (MacLaurin series) for the function f given above. Simplify your answer and write in $\sum$-notation.

(b) (4 points) Use the series from part (a) to find $f^{(8)}(0)$. You do not need to simplify your final answer.

(a) $2x \cdot f(x) = \sin\left(\frac{x}{3}\right)$

$$= \sum_{N=0}^{\infty} \frac{(-1)^N \left(\frac{x}{3}\right)^{2N+1}}{(2N+1)!} = \sum_{N=0}^{\infty} \frac{(-1)^N x^{2N+1}}{3^{2N+1}(2N+1)!}$$

$$f(x) = \frac{1}{2} \cdot \sum_{N=0}^{\infty} \frac{(-1)^N x^{2N}}{3^{2N+1}(2N+1)!} \quad (*)$$

(b) recall the Taylor series expansion:

$$f(x) = \sum_{N=0}^{\infty} \frac{f^{(N)}(0)}{N!} x^N \quad (**)$$

→ Find the coefficient of x^8 in $(*)$ and $(**)$:

in $(*)$: $2N = 8 \longrightarrow N = 4$

$$\frac{(-1)^4}{3^9 \cdot 9!} \overset{by\,(**)}{=} \frac{f^{(8)}(0)}{8!}$$

$$\longrightarrow f^{(8)}(0) = \frac{1 \cdot 8!}{3^9 \cdot 9!} = \frac{1}{3^9 \cdot 9} = \frac{1}{3^{11}}$$

4. (a) (12 points) Find the second degree Taylor Polynomial for the function $f(x) = x^{1/4}$ centered at $a = 1$.

(b) (8 points) Estimate $f(0.2)$ for your answer in part (a). Simplify your answer as far as you can without a calculator.

(c) (10 points) Find the maximum error in your approximation in part (b). Simplify as far as you can without a calculator. Recall: $|R_n(x)| \leq \max|f^{(n+1)}(c)| \frac{x-a}{(n+1)!}$.

(a) find $P_2(x) = \sum_{k=0}^{2} \frac{f^{(k)}(1)}{k!} (x-1)^k$ $(*)$

$\longrightarrow$ first find the derivatives in the formula:

$f^{(0)}(x) = f(x) = x^{1/4} \longrightarrow f^{(0)}(1) = 1$

$f^{(1)}(x) = \frac{1}{4x^{3/4}} \longrightarrow f^{(1)}(1) = \frac{1}{4}$

$f^{(2)}(x) = -\frac{3}{16x^{7/4}} \longrightarrow f^{(2)}(1) = -\frac{3}{16}$

$\longrightarrow P_2(x) = 1 + \frac{1}{4 \cdot 1!}(x-1) - \frac{3}{16 \cdot 2!}(x-1)^2$

$\quad = 1 + \frac{1}{4}(x-1) - \frac{3}{32}(x-1)^2$ $(**)$

(b) estimate $f(0.2) = f(\frac{1}{5})$

$\longrightarrow$ plug in $x = \frac{1}{5}$ into $(**)$ to find $P_2(\frac{1}{5})$

(c) How good/bad was our approx. in part (b)?

$\longrightarrow$ use the Taylor remainder formula:

$|R_N(x)| \leq \max_{|x-a| \leq c} |f^{(N+1)}(c)| \cdot \frac{|x-a|^{N+1}}{(N+1)!}$

$\longrightarrow$ here: take $N = 2$, $a = 1$, $x = \frac{1}{5}$

$\longrightarrow$ need $f^{(3)}(x)$:

recall from above that $f^{(2)}(x) = -\frac{3}{16x^{7/4}}$

$$f^{(8)}(x) = \frac{21}{64 x^{11/4}}$$

$\longrightarrow$ compute :

$$\underset{\frac{4}{5} \leq c}{Max} \frac{21}{64 \, c^{11/4}} = \frac{21}{64 \left(\frac{4}{5}\right)^{11/4}}$$

$\longrightarrow$ $\left| R_2\left(\frac{1}{5}\right) \right| \leq \dfrac{21}{64\left(\frac{4}{5}\right)^{11/4}} \cdot \dfrac{\left(4/5\right)^3}{3!}$

$$S = \sum_{k=2}^{\infty} \frac{\ln k}{k^5}$$

▸ (3 points) State the test and conditions. We will apply the integral test. We need the function to be positive, decreasing, and continuous. If $f(x) = \frac{\ln x}{x^5}$, then $f'(x) = \frac{x^4 - 5x^4 \ln x}{x^{10}}$ is negative when the numerator is negative. The numerator reduces to $x^4[1 - 5\ln x]$. We note that $1 - 5\ln x < 0$ when $\ln x > \frac{1}{5}$ or $x > e^{1/5} \approx 1.22$, so the function decreases when $x \geq 2$.

$$S = \sum_{k=2}^{\infty} \frac{\ln(k)}{k^5}$$

→ apply the integral test to check for convergence of S:

→ Conditions to apply the integral test (to $f(x) = \frac{\ln(x)}{x^5}$):

- $f(k)$ has to be positive for $k \geq 2$
- $f(x)$ has to be cts on $(2, \infty)$
- $f(x)$ needs to be decreasing:

Compute $f'(x) = \frac{d}{dx}\left[\frac{\ln(x)}{x^5}\right]$ $\left(\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}\right)$

$$= \frac{x^4 - \ln(x) \cdot 5 \cdot x^4}{x^{10}} \underbrace{< 0}_{\text{Need this to happen}}$$

↪ $1 - 5\ln(x) < 0$

↪ $\ln(x) > \frac{1}{5}$ ↪ $x > e^{1/5}$

(This happens when $x \geq 2$)

→ So we have checked the conditions, and we can apply the integral test

→ integral test tells us that

S converges exactly if

$\int_2^{\infty} f(x)dx$ converges (and diverges otherwise)

$$I = \int_2^\infty \frac{\ln(x)}{x^5}\,dx \qquad \left(\begin{array}{l} u\text{-sub:} \\ u = \ln(x), \quad \frac{1}{x^4} = e^{-4u} \\ du = \frac{dx}{x} \end{array} \right)$$

$$= \int_{\ln(2)}^\infty u\,e^{-4u}\,du \qquad \left(\begin{array}{l} \underline{IBP:} \\ \int u\,dv = uv - \int v\,du \end{array} \right)$$

$$\left[\begin{array}{ll} u = u & dv = e^{-4u}\,du \\ du = du & v = -\tfrac{1}{4}e^{-4u} \end{array} \right]$$

$$= -\frac{1}{4}u\,e^{-4u}\Big|_{\ln(2)}^\infty + \frac{1}{4}\int_{\ln(2)}^\infty e^{-4u}\,du$$

$$= -\frac{1}{4}u\,e^{-4u}\Big|_{\ln(2)}^\infty - \frac{1}{16}e^{-4u}\Big|_{\ln(2)}^\infty$$

$$= \lim_{x\to\infty} -\frac{1}{4}x\,e^{-4x} + \text{constant}$$

$$+ \lim_{x\to\infty} -\frac{1}{16}e^{-4x} + \text{constant}$$

$$= \text{constant}$$

$\longrightarrow$ So the integral I converges, and this shows that S converges.

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{(k^4+5)^{1/5}}(2x-4)^k$$

$$f(x) = \sum_{k=1}^{\infty} \frac{(2x-4)^k}{(k^4+5)^{1/5}}, \quad \text{find radius of conv } (R),$$
$$\text{IC}$$

$\longrightarrow$ write the series for $f(x)$ in "standard form", or in powers of $(x-c)$:

$$f(x) = \sum_{k=1}^{\infty} \frac{2^k}{(k^4+5)^{1/5}}(x-2)^k \quad \left(c=2, \right.$$
$$\left. a_k = \frac{2^k}{(k^4+5)^{1/5}} \right)$$

$\longrightarrow$ apply the ratio test:

$$L = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \frac{2^{k+1}}{2^k} \frac{\cancel{(k^4+5)^{1/5}}^{\,1}}{((k+1)^4+5)^{1/5}}$$

$$= 2$$

So that $R = \dfrac{1}{L} = \dfrac{1}{2}$

$\longrightarrow$ to find the IC:

$$|x-2| < R \iff -\frac{1}{2} < x-2 < \frac{1}{2}$$

$$\longrightarrow \frac{3}{2} < x < \frac{5}{2},$$

So the series definitely converges for all $x \in \left(\frac{3}{2}, \frac{5}{2}\right)$. We need to check convergence at the end points when $x=\frac{3}{2}, x=5/2$.

$$\longrightarrow f\left(\tfrac{3}{2}\right) = \sum_{k=1}^{\infty} \frac{2^k \left(-\tfrac{1}{2}\right)^k}{(k^4+5)^{1/5}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(k^4+5)^{1/5}}$$

apply the AST:

- check for absolute convergence:
  does $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k^4+5)^{1/5}} \longleftarrow a_k$ converge or not?

  <u>intuition</u>: looks "almost" like a p-series with $p = 4/5 \leq 1$, so we expect the series to diverge.

  <u>apply the LCT</u>: compare to $b_k = \dfrac{1}{k^{4/5}}$

  $$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^{4/5}}{(k^4+5)^{1/5}}$$

  $$= \lim_{k\to\infty} \frac{\cancel{k^{4/5}}^{1}}{\cancel{k^{4/5}}\left(1+\frac{5}{\cancel{k^4}}\right)^{1/5}} = 1$$

  So both series diverge.

- check for conditional convergence:
  $$\longrightarrow \lim_{k\to\infty} \frac{1}{(k^4+5)^{1/5}} = 0 \checkmark$$

  $\longrightarrow$ check that a_k is decreasing
  $$a_{k+1} = \frac{1}{((k+1)^4+5)^{1/5}} \leq a_k = \frac{1}{(k^4+5)^{1/5}} \checkmark$$

So we get conditional convergence.